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ELASTOSTATIC RESPONSE OF AN ECCENTRICALLY LOADED LAMINATED COMPOSITE PANEL WITH DISCRETE PERIODIC STIFFENERS

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ABSTRACT
Elastostatic response of a laminated composite panel is analyzed under the influence of periodic stiffeners. The discrete axial stiffeners are considered at the opposing longitudinal edges of a cross-ply laminated panel subjected to an eccentric axial tension. An efficient computational scheme is developed based on a new scalar function to analyze the present mixed-boundary-value stress problem of laminated composites. Results are claimed to be highly accurate and reasonable, which are however found to be in significant contrast with those of continuous as well as stiffener-free panels.

Keywords: Laminated Composite, Eccentric Loading, Periodic Stiffeners, Scalar Function, Numerical Solution.

1. INTRODUCTION
The analysis of composite structures has now become a key subject in the field of solid mechanics. Strength-to-weight ratio of a fiber reinforced composite material is usually higher than that of the corresponding isotropic material. The use of composite materials is gradually increasing day-by-day, especially, to satisfy the demand of light weight structures. On the other hand, the use of stiffeners, in general, is emphasized at the bounding surfaces of structural components primarily to enhance the stiffness of the surface in a particular direction. In the solution of stiffened structures, the physical condition of stiffeners is mathematically modeled in terms of a mixed mode of boundary conditions, for which one of the components of stress as well as displacement at a location of the boundary is assumed to be specified. As the earlier mathematical models of elasticity were inadequate in handling the mixed-boundary-value problems of elasticity, the practical stress problems, especially with stiffeners and boundary restraints, could hardly be solved with full satisfaction.

Elasticity problems are usually formulated either in terms of stress function or displacement parameters [1]. The shortcoming of the stress function approach is that it accepts boundary conditions only in terms of loadings, boundary restraints cannot be imposed on the function appropriately. Although analytical methods of solution give exact solution of stress field, they could not gain that much popularity in the field of stress analysis of structures, mainly because of inability of dealing with mixed mode of physical conditions as well as complex boundary shapes, which are however very common for the case of structural components in practice. Some of the examples of application of the stress function approach are cited here as reference [2-3]. Conway and Ithaca [4] first extended the stress function formulation in the form of Fourier integrals for the stress analysis of some ideal problems of orthotropic materials. Although elasticity problems were formulated long ago, exact solutions of practical stress problems are hardly available because of the inability to manage the physical shape and conditions imposed on them. In fact, appropriate management of boundary shapes and conditions is still remained as one of the obstacles to the reliable solution of practical problems.

Stress analysis of actual structures of composite materials is mainly handled by numerical methods, especially, by the finite element methods [5-7]. Although the finite element method relieved us from one of the major inability of modeling, *i.e.*, managing odd boundary shapes, reliable and accurate prediction of stresses, especially at the critical regions of surfaces of structural components, which is of utmost importance for reliable and economic design, is not always encountered in practical computation. The uncertainty associated with the finite element prediction of surface stresses has been pointed out by a number of researches in the field [8-10]. Recent research and developments in using the displacement potential boundary modeling approach [11, 12] have generated renewed interest in the field of both analytical and numerical solutions of stress problems orthotropic composite lamina [13-15]. Recently, *Ahmed et al*. [16] has proposed a new variable reduction approach to develop a single scalar function based computational scheme for the analysis of general anisotropic composite structures. The present paper extends the potential of the scalar function based computational scheme to the analysis of stresses and deformation of a laminated composite structure. A laminated panel subjected to an eccentric axial tension has been considered as the example problem, and the solution is obtained under the influence of periodic axial stiffeners along the two opposing longitudinal surfaces. It is mentioned that no serious attempt has been made so far in the literature that can provide reliable and accurate solution to the present mixed boundary value problem of laminated composites taking into account a large number of stress singularities in the solution.

2. MATHEMATICAL FORMULATION

For a symmetric laminated composite, the mid-plane strains are assumed to be equal to the global strains, as the effect of curvature of the laminate under in plane loading is usually neglected. For this case, the stress-strain relations under the plane stress condition are given by [17,18]

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = \frac{1}{h}\begin{bmatrix}A_{11} & A_{12} & 0\\ A_{12} & A_{22} & 0\\ 0 & 0 & A_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{Bmatrix} \tag{1}$$

where,

$$A_{11} = \sum_{k=1}^{n}\left[Q_{11}\cos^4\theta + 2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta\right]\left(h_k - h_{k-1}\right)$$

$$A_{12} = \sum_{k=1}^{n}\left[\left(Q_{11}+Q_{22}-4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta+\cos^4\theta\right)\right]\left(h_k - h_{k-1}\right)$$

$$A_{22} = \sum_{k=1}^{n}\left[Q_{11}\sin^4\theta + 2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta\right]\left(h_k - h_{k-1}\right)$$

$$A_{66} = \sum_{k=1}^{n}\left[\left(Q_{11}+Q_{22}-2Q_{12}-2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta+\cos^4\theta\right)\right]\left(h_k - h_{k-1}\right)$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\ Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},\ Q_{66} = G_{12}$$

h is the total thickness of the laminate, a and b are the panel length and width, respectively, E_1 and E_2 are the elastic moduli in lateral and longitudinal directions of each ply, respectively, ν_{12} and ν_{21} are the major and minor Poison's ratio, respectively; G_{12} is the in-plane shear modulus.

Following the procedure of a newly developed variable reduction scheme [16], and making use of the above constitutive relations (1), the single governing differential equation of equilibrium for the symmetric cross-ply laminated composites is obtained as follows:

$$\frac{\partial^4\psi}{\partial x^4} + \left[\frac{A_{22}}{A_{66}} - \frac{A_{12}^2}{A_{11}A_{66}} - \frac{2A_{12}}{A_{11}}\right]\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{A_{22}}{A_{11}}\frac{\partial^4\psi}{\partial y^4} = 0 \tag{2}$$

where, $\psi(x,y)$ is a scalar function of space variables [11,12], which is defined in terms of the two displacement components of plane elasticity, for the problems of laminated composites, as follows:

$$u_x(x,y) = \frac{\partial^2\psi}{\partial x\partial y} \tag{3}$$

$$u_y(x,y) = -\frac{1}{A_{12}+A_{66}}\left[A_{11}\frac{\partial^2\psi}{\partial x^2} + A_{66}\frac{\partial^2\psi}{\partial y^2}\right] \tag{4}$$

The expressions for the stress components for the symmetric cross-ply laminated composites, in terms of the scalar function, $\psi(x,y)$, are,

$$\sigma_{xx}(x,y) = \frac{A_{66}}{h\left(A_{12}+A_{66}\right)}\left[A_{11}\frac{\partial^3\psi}{\partial x^2\partial y} - A_{12}\frac{\partial^3\psi}{\partial y^3}\right] \tag{5}$$

$$\sigma_{yy}(x,y) = \frac{1}{h\left(A_{12}+A_{66}\right)}\left[\left(A_{12}^2 + A_{12}A_{66} - A_{11}A_{22}\right)\frac{\partial^3\psi}{\partial x^2\partial y} - A_{22}A_{66}\frac{\partial^3\psi}{\partial y^3}\right] \tag{6}$$

$$\sigma_{xy}(x,y) = \frac{A_{66}}{h\left(A_{12}+A_{66}\right)}\left[A_{12}\frac{\partial^3\psi}{\partial x\partial y^2} - A_{11}\frac{\partial^3\psi}{\partial x^3}\right] \tag{7}$$

3. NUMERICAL SOLUTION

The geometry and loading of a laminated composite panel subjected to an eccentric axial loading is shown in Fig. 1. The length, width and thickness of the panel are represented by a, b and h, respectively. The symmetric cross-ply laminate panel is composed of seven layers of identical thickness, the stacking sequence of which can be expressed as $[90/0/90/\bar{0}\,]_s$. The three-dimensional laminated panel is analyzed here considering it as a plane stress problem. The thickness of each ply is expressed as $h_k = \left(h_k - h_{k-1}\right)$ and thus the total laminate thickness becomes $h = nh_k$.

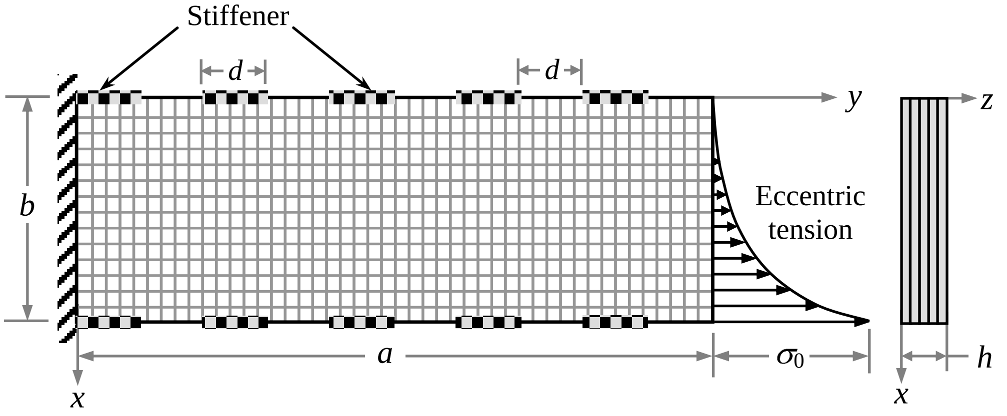

Fig. 1 Model of an eccentrically loaded laminated panel with periodic stiffeners

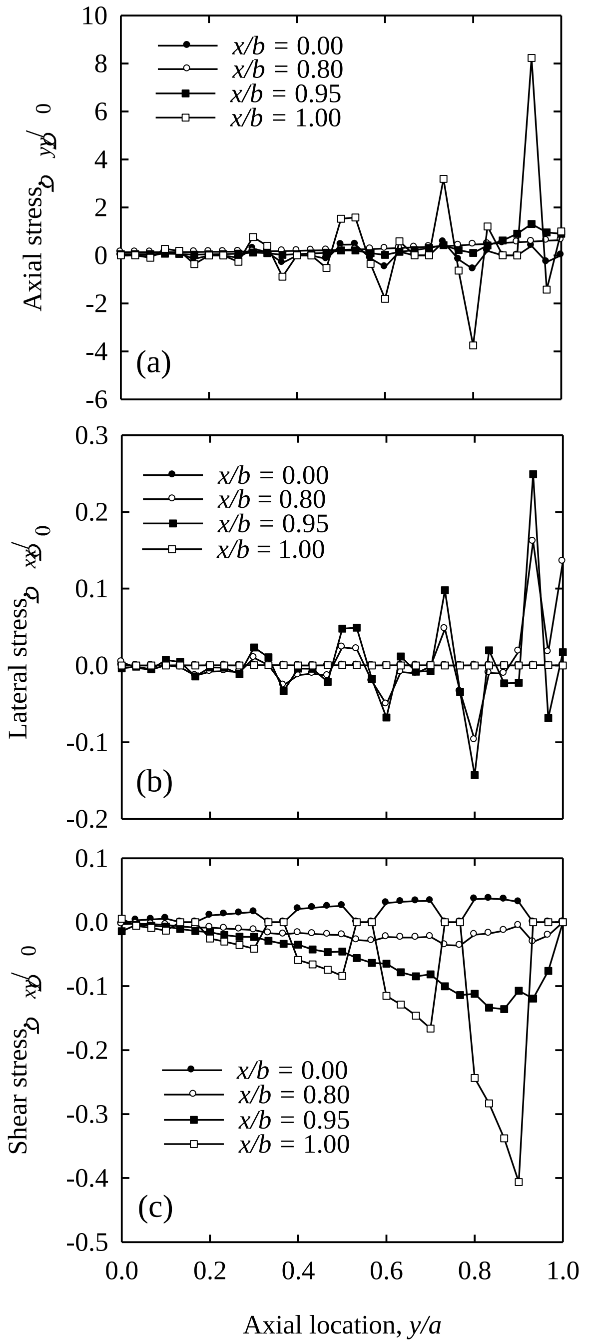

Fig. 2 Distributions of different stress components at different sections of the laminated panel: (a) axial, (b) lateral, and (c) shear stress

The left lateral end of the panel is kept rigidly fixed. The discrete periodic axial stiffeners are assumed to be situated along the two opposing longitudinal edges of the panel. The right lateral end of the panel is subjected to an eccentric axial tensile loading (parabolic distribution) of maximum intensity σ_0 = 3 MPa. The physical conditions considered along the bounding edges of the composite panel can be expressed mathematically as follows. The boundary conditions for the fixed supporting edge are taken as $u_x(x, 0) = u_y(x, 0) = 0$. The conditions of the two opposing longitudinal edges are basically of two types, one for the stiffeners and the other for the stiffener free regions. The stiffened regions are simulated by satisfying the conditions, $u_y(0, y) = \sigma_{xx}(0, y) = 0$, and that of free boundaries by $\sigma_{xx}(0, y) = \sigma_{xy}(0, y) = 0$. The parabolic axial tension at the right lateral end of the panel is simulated by $\sigma_{xy}(x, a) = 0$, and $\sigma_{yy}(x, a) = \sigma_0 (x/b)^2$.

The present computational scheme involves evaluation of a single variable $\psi(x, y)$ at the mesh points of a uniform rectangular mesh-network over a geometrically regular region. The governing differential equation (2), which is used to evaluate the function ψ at the internal mesh points of mid-plane of the laminate, can be expressed in its corresponding algebraic form using central difference operators. The complete finite-difference expression of the governing equation is given by

$$Z_1\{\psi(i-2, j) + \psi(i+2, j)\} - (4Z_1 + 2Z_2)\{\psi(i-1,j) + \psi(i+1, j)\} - (2Z_1 + 4Z_3)\{\psi(i, j+1) + \psi(i, j-1)\} + (6Z_1+4Z_2+6Z_3)\psi(i, j) + Z_2\{\psi(i-1, j-1) + \psi(i-1, j+1) + \psi(i+1, j-1) + \psi(i+1, j+1)\} + Z_3\{\psi(i, j-2) + \psi(i, j+2)\} = 0 \quad (8)$$

where, $Z_1 = \dfrac{1}{h_x^{\,4}}$, $Z_2 = \left(\dfrac{A_{22}}{A_{66}} - \dfrac{A_{12}^{\,2}}{A_{11}A_{66}} - \dfrac{2A_{12}}{A_{11}}\right)\dfrac{1}{R^2 h_x^{\,4}}$,

$Z_3 = \dfrac{A_{22}}{A_{11}}\dfrac{1}{R^4 h_x^{\,4}}$, $R = \dfrac{k_y}{h_x}$

From equation (8), it is thus clear that, application of the governing equation involves a total of 13 neighboring mesh points, and when this point of application (i, j) becomes an immediate neighbor to the physical boundary, the equation will involve mesh points not only interior but also exterior to the physical boundary.

As the differential equations associated with the boundary conditions contain second- and third-order partial derivatives of the function ψ, different versions of the finite-difference expressions like forward, backward and central have been adopted in a combined form to generate the difference equations for the boundary conditions. It is noted here that the order of local truncation error has been kept the same for all the expressions developed, that is, $O(h^2)$. Four different sets of finite-difference expressions for each of the boundary conditions are developed for nodal points in different regions of boundary.

4. RESULTS AND DISCUSSIONS

In this section, results of the numerical solution of the laminated panel having an aspect ratio (a/b) of 2.5 are presented mainly in the form of graphs. Although the method can be applied to any composite material, the boron/epoxy composite is chosen as an example in present study. The effective mechanical properties of boron/epoxy laminate are listed in Table 1. In order to make the results non-dimensional, the stresses are expressed as the ratio of the developed stress to the maximum intensity of the eccentric loading applied. The solutions are obtained here for the laminated panel having five equally-spaced discrete stiffeners along the two opposing longitudinal edges, which, in turn, makes the length of each stiffener, d/a = 0.1 (see Fig. 1).

Table 1: Mechanical properties of boron/epoxy composite material

Composite	Property	Value
Boron/ Epoxy	E_1 (10^3 MPa)	204.00
	E_2 (10^3 MPa)	18.50
	G_{12} (10^3 MPa)	5.59
	ν_{12}	0.23

Figure 2 illustrates the distributions of normalized stress components with respect to the axial location of the panel, for different lateral sections. It is observed from Fig. 2(a) that the highest axial stress develops at the bottom longitudinal surface of the panel. The magnitude of the stress is found to increase with the increase of axial distance from the left lateral end. The maximum intensity of the axial stress in the laminated panel is found to assume a value as high as eight times of σ_0, the location of which is at the region of the last stiffener near the right lateral end. Overall, the state of the axial stress within the panel is identified to be the most significant when compared to the other components of stress. Figure 2(b) shows the distributions of the lateral stress component within the panel. Both the two opposing longitudinal stiffened surfaces of the panel (x/b = 0, 1) are found to be completely free from lateral stresses. Though the stiffeners are present, no lateral stresses are developed at these sections, which is in good agreement with the physical characteristics of the panel as well as the applied loading. Maximum intensity of the lateral stress is found to develop at section x/b = 0.95, which is after end of the stiffening region near the loaded boundary. Fluctuation of stresses has been occurred due to the existence of discrete periodic stiffeners. However, the fluctuation is found to decrease gradually as we move towards the top boundary. Both axial and lateral stresses are found to assume insignificant values near the regions of the left supporting end.

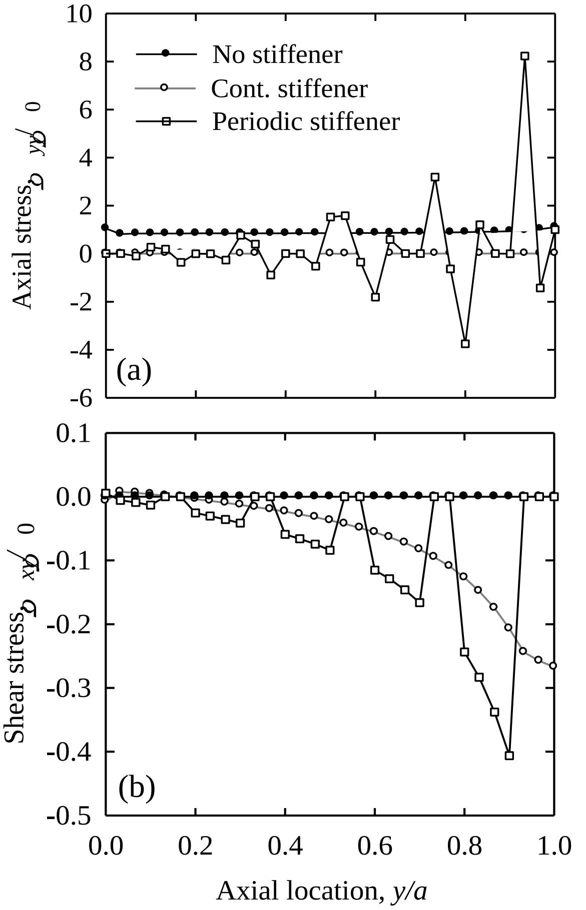

Fig. 3 Distribution of stresses along the bottom edge of the laminated panel with different conditions of stiffeners

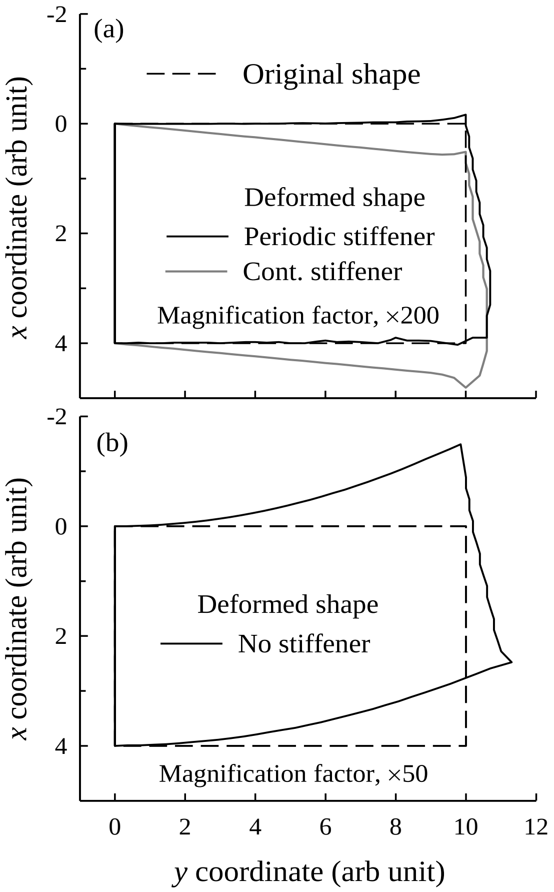

Fig. 4 Deformed shape of the laminated panel with different conditions of stiffeners

The distribution of shearing stress at different sections of the panel is presented in Fig. 2(c). The maximum and minimum shear stresses are found to occur at the top and bottom bounding surfaces, respectively. More specifically, the shear stress at the stiffened region of the bottom surface assumes negative value, which is found to increase gradually as one moves from the left end to the right lateral end For sections, x/b = 0 and 1, the predicted shear stresses are found to be zero at the no stiffener regions, which is again in good conformity with the physical characteristics of the problem. Overall, the magnitude of the shear stress is found to be higher than those of lateral stresses developed in the panel.

Figure 3 illustrates the comparison of distribution of normalized stress components for the different stiffening effect on the bottom boundary (x/b = 1.0) of the panel. The results basically compare the state of stresses along the periodic stiffeners with those associated with the corresponding cases of continuous stiffener and stiffener free panels. When the surfaces of the panel are free from stiffeners, normalized axial stress is found to be almost constant and equal to the maximum applied tension at the right lateral end. But no axial stress is found to develop along the bottom surface when a continuous stiffener is attached along both the opposing longitudinal surfaces. However, the present periodic stiffener changes the state of the stress along the boundary significantly from those of the continuous and no stiffeners, as shown in the figure. Gradually increasing stresses with both positive and negative magnitudes are obtained with an alternative fashion for the case of periodic stiffener. Figure 3(b) describes the comparison of shear stress distributions for the three cases of stiffeners mentioned above. For the stiffener free panel, the predicted shear stress at the bottom surface is found to be zero, which is again in good agreement with the physical characteristics of the panel. For the case of continuous stiffeners, the distribution assumes a quadratic/cubic form, which is gradually increasing from left to right. A somewhat similar distribution is obtained for the case of present periodic stiffener, which is however much higher in magnitude compared to the case of continuous stiffener. As expected, the stiffener free regions of the boundary are found to be free from shearing stresses. The maximum shear stress developed at the bottom surface of the panel with periodic stiffener is found to be nearly 50% higher than that observed for the case of continuous stiffener.

Figure 4 illustrates the deformed shapes of the laminated panel obtained with different cases of stiffener used along the two opposing longitudinal surfaces. Overall, it is revealed from the results that axial stiffeners have significant influence on the deformation pattern of the panel, particularly for the present eccentric axial loading. The axial stiffeners at the surfaces, in general, cause the overall deformation of the panel to assume much smaller magnitude, which can be easily realized when compared to the case of no stiffener, as shown in Fig. 4. It is interesting to note that, when the longitudinal edges of the panel are made free from stiffeners, the present eccentric axial loading causes the deformation of the panel somewhat similar to the case of subjecting an anti-clockwise moment at the right lateral end, where the lateral component of displacement is found to be more prominent than the axial component. The above bending-like deformation pattern is not observed for the case of discrete periodic stiffeners. The deformation pattern rather conforms to the general characteristic of the model of the panel, as the axial deformation is found to be highest at the lower surface and almost zero at the upper surface of the panel (see Fig. 4(a)). A critical analysis however shows that the panel with periodic stiffeners experiences a bending-like deformation, somewhat similar to that of stiffener-free panel, only for the small stiffener-free region adjacent to the right lateral end. In order to support the present eccentric axial tension, the panel is found to follow a deformation pattern completely different from those of periodic and no stiffeners, when the two opposing longitudinal surfaces are subjected to full-length continuous stiffeners. The obtained deformed state of the panel is shown in Fig. 4(a), in which the right extreme corner of the bottom surface is found to experience the contribution of both the lateral and axial displacements in the positive direction. As a result, to satisfy the requirement of the applied loading as well as the given conditions on the opposing surfaces, the top-right corner of the panel is found to experience only the contribution of lateral displacement in the positive direction.

5. CONCLUTIONS

A new scalar function based computational scheme has been developed to analyze the elastostatic response of laminated composite panel subjected to an eccentric axial tension, under the influence of discrete periodic stiffeners. No appropriate analytical approach is available in the literature, which can provide explicit information about the actual stresses, especially at the critical regions of supports and stiffeners of the laminated panel. Both the qualitative and quantitative results of the present mixed-boundary-value stress problem of laminated composites establish the soundness as well as appropriateness of the single function approach. The periodic stiffeners are found to have significant influence on the deformed shape of the panel, especially when compared with those of continuous and stiffener-free panels. Moreover, both the axial and shear stresses along the edges of the panel are found to assume much higher magnitudes from those of continuous and stiffener-free panels. The solutions are claimed to be highly accurate and rational for the entire region of interest, either at or away from the bounding edges of the composite panel. It is thus expected that the present data would be considered as a valuable design guide for laminated panels with discrete local stiffeners.

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7. NOMENCLATURE

Symbol	Meaning	Unit
ψ	Scalar function	-
E_1	Elastic modulus in lateral direction	MPa
E_2	Elastic modulus in axial direction	MPa
G_{12}	Shear modulus of elasticity	MPa
ν_{12}	Major Poisson's ratio	
σ_{xx}	Lateral stress	MPa
σ_{yy}	Axial stress	MPa
σ_{xy}	Shear stress	MPa
σ_0	Maximum intensity of the axial eccentric loading	MPa
u_x	Lateral displacement	m
u_y	Axial displacement	m
a	Length of the panel	m
b	Breadth of the panel	m
h	Thickness of laminated panel	m
n	Number of ply in laminate	-
h_x, k_y	Mesh lengths in x- and y-directions	m

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